Public Economics
Overlapping generation economics
Production economy

Luca Spataro

Hypotheses

- Discrete time (without uncertainty)

1) Individuals

- Two period lived individuals whose generations overlap
- Population grows at the exogenous rate n
- Individuals maximize their own lifetime utility over consumption, taking prices as given (expectations later on)
- Individuals work in the first period (labour supply normalized to 1) receiving a wage w and retire in the second period.
- Hence they have to save out of their wages for consuming in the second period
- Income, consumption and savings occur at the beginning of the period (convention)

2) Firms

- Firms run their business in a perfectly competitive setting and maximize their profits
- Hire capital and labour and pay them according to their marginal productivity

3) Markets

- There are three markets: labour market, goods market, and capital market (capital is a production good, like corn).
- Capital is offered by young households to firms and enters the production process the next period (one period delay in production), Hence it is given back with interests in the next period (when lenders are old)

Individuals

- Individuals of generation t maximize a lifetime utility function defined over consumption in the two periods and assuemed additive and separable:
- $U_t=u(c_{1t})+\beta u(c_{2t+1})$,

with $0<\beta<1$ the intertemporal discount factor and

H2: $u'>0$, $u''<0$+ $\lim(c\to 0)u'=\infty$.

Individuals (cont'd)

- Budget constraint (in real terms, or assuming p_t and p_{t+1}=1)

$w_t=c_{1t}+s_t$
$s_t(1+r(e)_{t+1})=c_{2t+1}$

where s_t is savings and $r(e)_{t+1}$ is expected return on savings and $R(e)_{t+1} \equiv 1+r(e)_{t+1}$. The lifetime version of the individual's budget constraint is simply:

$c_{1t}+c_{2t+1}/(1+r_{t+1})=w_t$

- Hence, the individual's problem is

max U=max $(u(w_t-s_t)+ \beta u(s_t(1+r(e)_{t+1}))$

with respect to s_t which yields:

$u'(c_{1t})/(u'(c_{2t+1}))= \beta(1+r(e)_{t+1})$

which is the so called Euler's equation.

- Note that if individuals are rational, $r(e)_{t+1}=r_{t+1}$, while if they are myopic, then we will assume that $r(e)_{t+1}=r_t$

Properties of demand functions

The solution provides demand functions $c_1=c_1(w_t,r_{t+1})$, $c_2=c_2(w_t,r_{t+1})$ and the saving function $s_t=s(w_t,r_{t+1})$.

1) Under separability and concavity of the utility function, consumption is normal and the marginal propensity to save is positive and less than 1.

Proof

- By differentiating Euler's equation and the lifetime budget constraint with respect to c_1 and w we get:

$$\frac{\partial c_1}{\partial w} = \frac{1}{1+\frac{1}{\beta(1+r)^2}\frac{u_1''}{u_2''}} = \frac{1}{1+\frac{1}{\beta(1+r)^2}} \in (0,1)$$

- And thus:

$$\frac{\partial s}{\partial w} = 1 - \frac{\partial c_1}{\partial w} \in (0,1) \qquad \frac{\partial c_2}{\partial w} = (1+r_{t+1})\frac{\partial s}{\partial w} > 0$$

The role of the interest rate

- Let us use the implicit function theorem.
- (see De la Croix-Michel 2002, p.311).

Take the FOC: $h \equiv u'(s) + \beta(1+r_{t+1})u'(s(1+r_{t+1})) = 0.$

Then

$$\frac{\partial s}{\partial r} = -\frac{h_r}{h_s} = -\frac{\beta u' + \beta s(1+r)u''}{u'' + \beta(1+r)^2 u''} = \frac{-u'}{u''}\frac{1-\frac{u''c_2}{u'}}{\frac{1}{\beta(1+r)^2}+1}. \text{ Hence}$$

$$sign\left(\frac{\partial s}{\partial r}\right) = sign\left(1-\frac{u''c_2}{u'}\right) = sign\left(1-\frac{1}{\sigma}\right)$$

where σ is the relative risk aversion coefficient (or the elasticity of marginal utility).

Therefore : $s_r \underset{<}{\overset{>}{-}} 0$ iff $\sigma \underset{<}{\overset{>}{-}} 1.$

(Interpretation in terms of substitution and income effect)

Firms

- Assume F(K,L) is Constant returns to scale
- Firms maximize current profits π_t

$$\max \pi_t = F(K_t, L_t) - w_t L_t - r_t K_t - \delta K_t,$$

where δ is depreciation of capital.

FOCS imply

$$\frac{\partial F}{\partial K_t} = (r_t + \delta),$$

$$\frac{\partial F}{\partial L_t} = w_t$$

Feasibility constraint

- The feasibility constraint is an accounting identity, whereby:
- $Y_t \equiv F(K_t, L_t) = C_t + I_t$

Intertemporal equilibrium

- It is a set of prices (w and r), allocations (consumption, labour supply, capital) and expectations (r(e)) which implement a competitive equilibrium in each period (temporary equilibrium).
- A temporary equilibrium is a set of prices and allocations that clear all markets and satisfy the feasibility contraint, given past history and price expectations.
- Note that at time t only two markets are open: labor and goods markets, in that the physical capital is already installed and investment I_t results from the decision of the young individuals in t-1.

Intertemporal equilibrium

- Given an initial capital stock $k_0=K_0/N_{-1}$, an inter-temporal equilibrium with perfect foresight is a sequence of temporary equilibria that satisfies for all $t\geq 0$ the condition:
- $(1+n)k_{t+1}=s(w_t(k_t),f(k_{t+1}))$

Proof

Suppose that $K_0 = s_{-1}N_{-1}$, that is at the beginning of the economy the installed capital belongs to the old and is determined by their saving decisions.

Given the feasibility constraint:

$F_t = C_t + I_t = c_{1t}N_t + c_{2t}N_{t-1} + K_{t+1} - K_t(1-\delta).$

By substituting for the individual s budget constraint s and demand functions we get:

$(i) \quad F_t = (w_t - s_t)N_t + (1+r_t)s_t N_{t-1} + K_{t+1} - K_t(1-\delta).$

Let us now exploit Euler's theorem:

$F_t = F_{k_t}K_t + F_{L_t}L_t$

and substitute for equilibrium conditions on the production input markets:

(ii) $F_t = (r_t + \delta)K_t + w_t L_t$

and equate it with the equation (i), such that

$(r_t + \delta)K_t + w_t L_t = (w_t - s_t)N_t + (1+r_t)s_t N_{t-1} + K_{t+1} - K_t(1-\delta).$

By exploiting the equilibrium in the labour market, such that $N_t = L_t$:

$K_{t+1} = (1+r_t)(K_t - s_{t-1}N_{t-1}) + s_t N_t.$

Since at $t = 0$ we have $K_0 = s_{-1}N_{-1}$, as for period $t = 1$ we get $K_1 = s_0 N_0$. Hence, recursively we obtain:

$K_{t+1} = s_t N_t$, and in capital intensity form:

$(1+n)k_{t+1} = s_t\left(w(k_t), r(e)_{t+1}\right).$

Finally, under rational expectations $r(e)_{t+1} = r_{t+1} = f'(k_{t+1}) - \delta$, such that

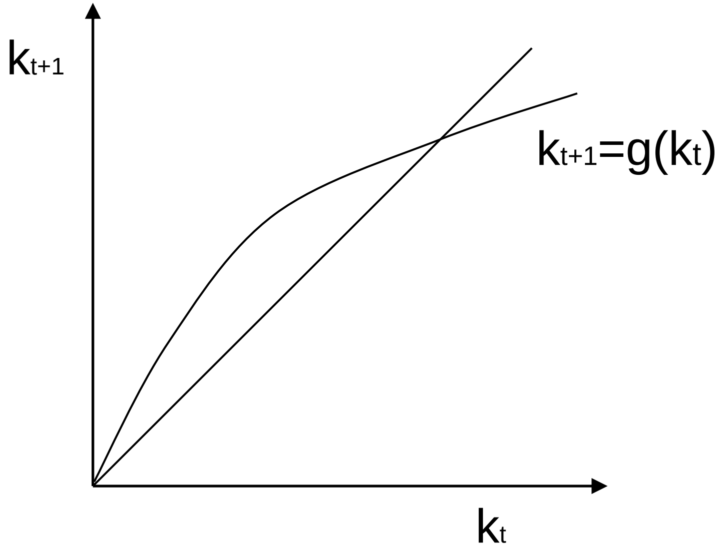
$(1+n)k_{t+1} = s_t\left(w(k_t), f'(k_{t+1})\right).$

Steady state

Definition

A steady state capital $\bar{k}$ is a level of capital such that $\bar{k} = g(\bar{k}) = s(\bar{k})/(1+n)$.

That is, $\bar{k}$ is the solution of the difference equation g.



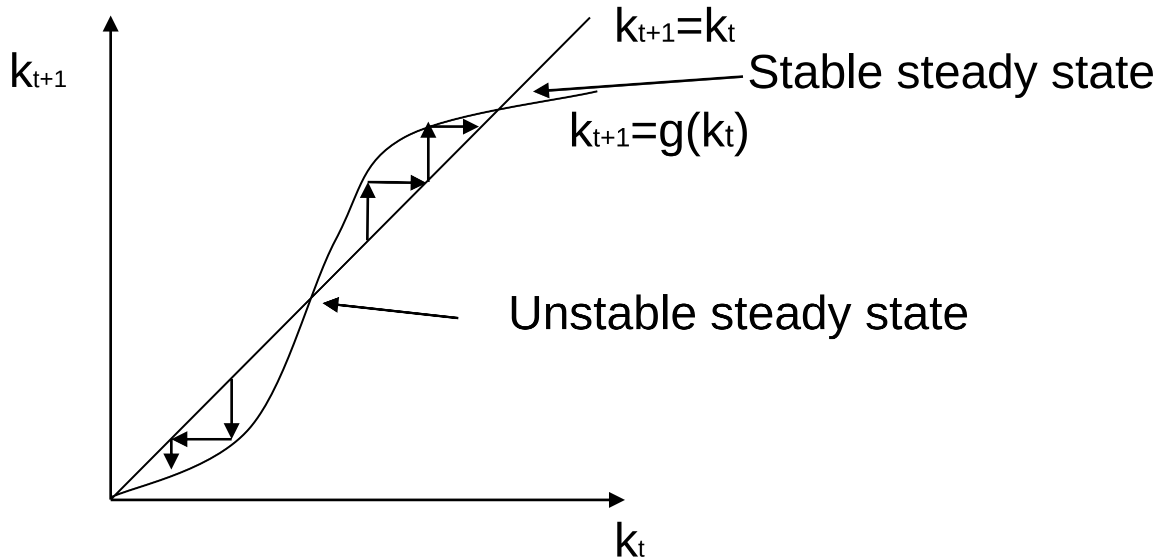
Steady state

Definition

A steady state capital $\bar{k}$ is locally stable if there exists $\varepsilon > 0$ such that for any k_0 in R_+ which verifies $|k_0 - \bar{k}| < \varepsilon$, the dynamics described by $k_{t+1} = g(k_t)$ with intial capital stock k_0 converge to $\bar{k}$.

Put it differently:

at the steady state $\left|\frac{dk_{t+1}}{dk_t}\right| < 1.$



By totally differentiating $(1+n)k_{t+1}=s(w_t(k_t),f(k_{t+1}))$ w.r.t. k_{t+1} and k_t we get that $\frac{dk_{t+1}}{dk_t} = -\frac{s_w f''}{1+n-s_r f''} < 1$. Recall that f''<0, s_w>0, $s_r > 0$ iff $\sigma > 1$.

Under these assumption the locus has 1 stable nonzero steady state

References

- De la Croix and Michel (2002), Chapter 1.