Public Economics
Welfare analysis in OLG models

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Golden Rules

- Different ways of defining the optimal level of capital:

A) the level that maximizes per capita consumption in the steady state (Golden Rule)

B) The level that maximizes a social welfare function (Modified Golden Rule)

Golden Rule

- Suppose that the objective function of a central planner is to maximize total consumption:
- $Max\ (c_{1t}N_t+c_{2t}N_{t-1})$
- Which in per capita terms is
- $c_{1t}+c_{2t}/(1+n)$
- At the steady state it can be written as:
- $c_1+c_2/(1+n)$

Golden Rule

- Recall that the economy's resource constraint implies

$$F_t = C_t + I_t = c_{1t}N_t + c_{2t}N_{t-1} + K_{t+1} - K_t(1-\delta).$$

- Which, in per-capita terms is

$$f_t = c_{1t} + \frac{c_{2t}}{1+n} + (1+n)k_{t+1} - k_t(1-\delta).$$

- Hence, maximizing total consumption means to maximize the function (suppose δ=0 for simplicity)

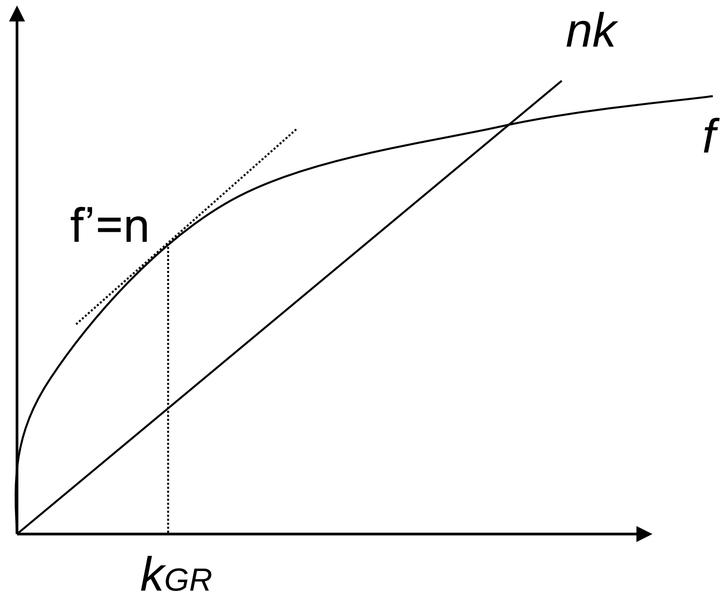
$$c_{1t} + \frac{c_{2t}}{1+n} = f_t - (1+n)k_{t+1} + k_t.$$

Golden Rule

- At the steady state

$$\max\ f - nk$$

- Graphically



Golden Rule

- At the steady state

$$\max f - nk$$

- The maximization provides the following result:

$$f' = n$$

- which is called Golden Rule
- Recall that the competitive equilibrium implies

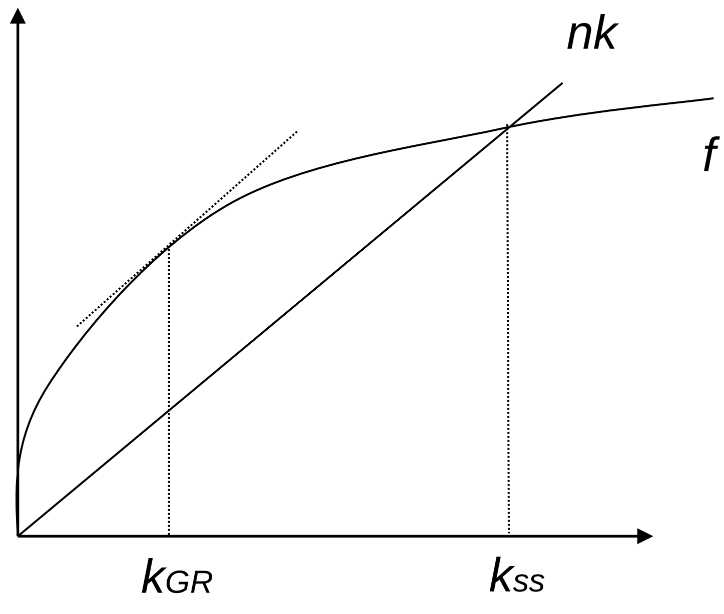
$$f' = r$$

- It follows that the competitive equilibrium will be optimal only if r=n. However, typically, this will not be the case.
- Motivation: individuals take r as exogenous. Only by coordinating trades they will implement an equilibrium which is also optimal from a social point of view.

Golden Rule

$$\max\ f - nk$$

- Graphical illustration



Sufficient condition for the existence of a positive k_{GR} *is* $f'(0)>(n)>f'(\infty)$

Mofidied Golden Rule

- Suppose that a policymaker aims at maximizing a social welfare function of the form:

$$\sum_{t=0}^{\infty} \gamma^t U(c_{1t}, c_{2t+1})$$

- Where $\gamma \in (0,1)$ is a weight attached to each generation. In each period the problem faces the constraint of the economy's resources

$$f(k_t) = c_{1t} + \frac{c_{2t}}{1+n} + (1+n)k_{t+1} - k_t \quad \forall t \geq 0$$

Modified Golden Rule

- Hence, the problem can be written as:

$$\max\left[U(c_{10}, c_{21}) + \gamma U(c_{11}, c_{22}) + \ldots + \gamma^{t-1}U(c_{1t-1}, c_{2t}) + \gamma^{t}U(c_{1t}, c_{2t+1}) + \ldots\right]$$

- Moreover, by substituting c_{1t-1} and c_{1t} from the resource constraint we get

$$\max\left[\ldots + \gamma^{t-1}U\left(f(k_{t-1}) - \frac{c_{2t-1}}{1+n} + (1+n)k_t + k_{t-1}, c_{2t}\right) + \gamma^{t}U\left(f(k_t) - \frac{c_{2t}}{1+n} - (1+n)k_{t+1} + k_t, c_{2t+1}\right) + \ldots\right]$$

- which provides the following FOCS:

$$w.r.t.\ \mathrm{c}_{2\mathrm{t}} : \gamma^{t-1}U_2 - \gamma^{t}\frac{U_1}{1+n} = 0 \Rightarrow \frac{U_1}{U_2} = \frac{1}{\gamma}(1+n)$$

$$w.r.t.\ \mathrm{k}_{\mathrm{t}} : (1+\mathrm{n})\gamma^{t-1}U_1 - \gamma^{t}U_1\left(1 + f'_{k_{t+1}}\right) = 0 \Rightarrow \text{(at the steady state)}: (1 + f') = \frac{1}{\gamma}(1+n)$$

- The first rule represents optimal allocation of consumption among young and old, while the latter represents optimal allocation of consumption through time.
- The competitive equilibrium will be optimal only if $r = \frac{1}{\gamma}(1+n) - 1$.

However, typically, this will not be the case.

Example

Take the following utility function (Diamond 1965)

$$U = \beta \log c_{1t} + (1-\beta)\log c_{2t+1}$$

And CD production function

$$f = Ak^{\alpha}$$

Then the steady state interest rate is

$$r = \frac{\alpha(1+n)}{(1-\beta)(1-\alpha)}$$

Which will be equal to the Golden rule level only if

$$r = n \Leftrightarrow n = \frac{\alpha}{(1-\beta)(1-\alpha)-\alpha}$$

Hence, the Golden rule economy will be the exception rather than the norm.

Comments

- 1) If $k_{SS} > k_{GR}$, then $r < n$.
- 2) If $k_{SS} < k_{GR}$, then $r > n$.

1) The economy is “over-accumulating” (“dynamic inefficiency”). By consuming some of the existing capital stock and saving less (i.e. increasing r) both the current generations and future generations would be better off (Pareto improvement).
2) The economy is “under-accumulatin”g (case of dynamic efficiency). Future generations would be better off by increasing the stock of capital (reducing r), which would increase future consumption. However, current generations should give up some of their consumption in order to increase the stock of capital. No Pareto improvement is available.

Policy implications

- Redistribution among generations
- Social security

Social Security

- Suppose a social security program is introduced, such that the individual budget constraint becomes:

$$c_{1t} + s_t = w_t - \tau_t$$

$$c_{2t+1} = p_{t+1} + s_t\left(1 + r_{t+1}\right)$$

- Where τ_t is the lump sum contribution and p_{t+1} is the pension.

Two models of social security

- Fully Funded (FF): $p_{t+1} = (1 + r_{t+1})\tau_t$
- Pay As you go (PAYG): $p_t N_t = \tau_{t+1} N_{t+1} \Rightarrow p_t = \tau_{t+1}(1+n)$

- Fully funded:
- 1) By substituting into the individual's budget constraint we get that

$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t - \tau_t + \frac{p_{t+1}}{1+r_{t+1}} = w_t$$

- The budget constraint does not vary (and so consumption)
- Moreover, at the equilibrium we have

$$\tau_t + s_t = (1+n)k_{t+1}$$

- Hence, neither the steady state capital varies: the increase in compulsory savings is offset by the decrease in private, voluntary savings in a one to one proportion.

PAYG Social Security (partial equilibrium)

- 1) By substituting into the individual's budget constraint we get that

$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t - \tau_t + \frac{p_{t+1}}{1+r_{t+1}} = w_t + \tau_t \frac{n - r_{t+1}}{1+r_{t+1}} \gtrless w_t \Leftrightarrow n - r_{t+1} \gtrless 0$$

- This is called "wealth effect", which affects consumption and savings. In fact, by differentiating Euler's equation and the budget constraint, for given prices, we get:

$$\frac{\partial c_1}{\partial \tau_t} = \frac{(1+r_{t+1})(n-r_{t+1})U_2^{"}}{U_1^{"} + U_2^{"}(1+r_{t+1})^2} \gtrless 0 \Leftrightarrow n \gtrless r_{t+1}$$

- Moreover, we get that the increase in compulsory saving ($s=w-c_1-\tau$) is offset by the decrease in private, voluntary savings in a more (less) then one to one proportion.

$$\frac{\partial s}{\partial \tau_t} = -1 - \frac{\partial c}{\partial \tau_t} \lessgtr -1 \Leftrightarrow n \gtrless r_{t+1}$$

PAYG Social Security

- However, also prices change. By reckoning that:

$$k_{t+1} = \frac{1}{1+n} s\left(w_t - \tau_t, r_{t+1}\right)$$

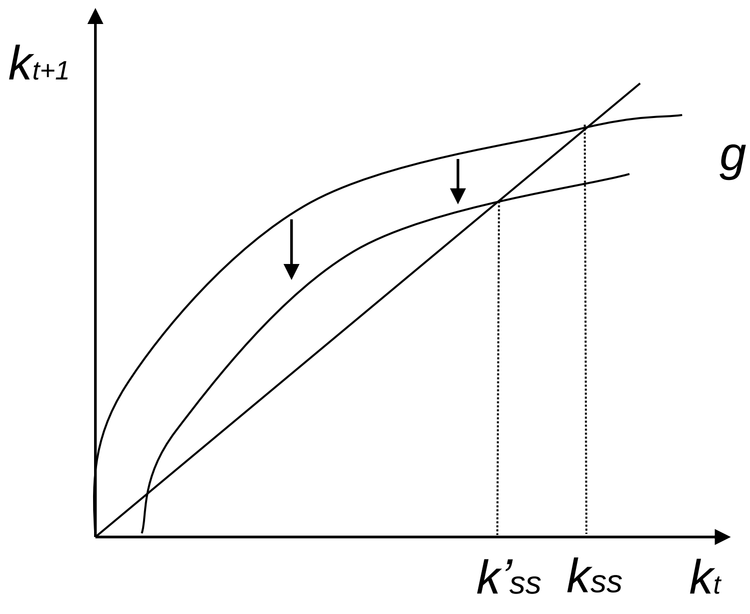
- And totally differentiating the latter equation at the steady state

$$\frac{dk}{d\tau} = \frac{1}{1+n}\left[s_w \frac{dw}{dk}\frac{dk}{d\tau} - s_w + s_r \frac{dr}{dk}\frac{dk}{d\tau}\right] \Rightarrow \frac{dk}{d\tau} = -\frac{s_w}{1+n+s_w kf'' - s_r f''} < 0$$

under stability of the steady state, i.e.:

$$\left|\frac{dk_{t+1}}{dk_t}\right| = \left|-\frac{s_w kf''}{1+n-s_r f''}\right| < 1$$

PAYG Social Security



Welfare implications (steady state)

Note that, since capital reduces, the interest rate increases (dr/dtau>0). As for welfare implications, let us differentiate the individual budget constraint

$$\frac{dc_1}{d\tau}+\frac{dc_2}{d\tau}\frac{1}{1+r}-\frac{c_2}{(1+r)^2}\frac{dr}{d\tau}=\frac{dw}{d\tau}+\frac{n-r}{1+r}-\frac{\tau(1+n)}{(1+r)^2}\frac{dr}{d\tau}$$

Let us differentiate total utility and exploit Euler's equation (envelop theorem, whereby $U_1/U_2=1+r$)

$$\frac{dU}{d\tau}=U_1\frac{dc_1}{d\tau}+U_2\frac{dc_2}{d\tau}=U_1\left(\frac{dc_1}{d\tau}+\frac{1}{1+r}\frac{dc_2}{d\tau}\right)$$

And substitute from eq. above

$$\frac{dU}{d\tau}=U_1\left(\frac{c_2}{(1+r)^2}\frac{dr}{d\tau}+\frac{dw}{d\tau}+\frac{n-r}{1+r}-\frac{\tau(1+n)}{(1+r)^2}\frac{dr}{d\tau}\right)$$

Reckoning that $\frac{c_2}{1+r}=s+\frac{p}{1+r}=k(1+n)+\tau\frac{1+n}{1+r}$ and that $\frac{dw}{d\tau}=\frac{dw}{dk}\frac{dk}{dr}\frac{dr}{d\tau}=-k\frac{dr}{d\tau}$ we get

$$\frac{dU}{d\tau}=U_1\left[\frac{k(1+n)}{(1+r)}\frac{dr}{d\tau}+\tau\frac{1+n}{(1+r)^2}\frac{dr}{d\tau}-k\frac{dr}{d\tau}+\frac{n-r}{1+r}-\frac{\tau(1+n)}{(1+r)^2}\frac{dr}{d\tau}\right]=\frac{(n-r)}{(1+r)}U_1\left[1+k\frac{dr}{d\tau}\right]\mathop{\gtrless}0\Leftrightarrow n\mathop{\gtrless}r$$

Comments

- Insofar as $n>r$, (overaccumulation) it is convenient to increase pensions (which are net wealth for individuals), consumption and crowding out savings/capital, up to the point in which $n=r$. After that point no Pareto improvement is possible.
- If $n<r$, it is convenient to reduce Social Security, although the first generations will be paying the cost for the improvement of the steady state welfare.

References

- Myles (1995): Public Economics, chapter. 13 (for social security)
- Social Security: Blanchard-Fisher (1989), chapter 3.2